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OPTIMAL CONTROL OF CERTAIN QUASILINEAR STOCHASTIC SYSTEMS

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The problem of optimal control of quasilinear systems in the presence of external random while noise-type perturbations is considered. Consecutive approximations to the optimal control are obtained and the errors along the trajectory and the optimal functional are estimated.

1. A number of papers in the field of optimal control of stochastic systems which have recently appeared deal with the study of controlled systems containing small terms. This can be explained, in particular, by the fact that although the basic formulations of the problems of stochastic control have been known for considerable time [1, 2], however conclusive results could only be obtained for the linear systems and a quadratic functional. A problem arises of constructing an approximate optimal control by expansion in the terms of a small parameter. For the case when the external perturbations are of low intensity, i.e. when a specified controlled system plays the part of the generating system, the problem of synthesizing an approximate control is dealt with in [3—5] where it is assumed that the solution of the problem is known, and has been obtained in the form of a synthesis.

Another approach to the problem of approximate synthesis of an optimal control is also possible, and is developed below. In it the role of the generating system is played by a linear controlled stochastic system with a quadratic criterion, for which the optimal control can be obtained in the explicit, analytic form. Speaking more accurately, in the present paper we assume that the controlled system has the form

$$x^{\cdot}(t) = A(t)\,x(t) + B(t)\,u(t) + \varepsilon f(t, x(t)) + \sigma(t)\,\xi^{\cdot}(t), \quad 0 \leqslant t \leqslant T, \qquad (1.1)$$
$$x(0) = a_0$$

Here the vector of the phase coordinates $x(t)$ belongs to the n-dimensional Euclidean space E_n, the control $u \in E_e$, the Wiener process $\xi \in E_n$, the matrices A, B and σ are given and have measurable elements on the segment $[0, T]$, the small parameter $\varepsilon \geqslant 0$, the constant $T \in [0, \infty)$ and the vector $a_0 \in E_n$.

The Wiener process $\xi(t)$ is normalized by the conditions

$$\xi(0) = 0, \qquad M\xi(t) = 0, \qquad M\xi_i^2(t) = t$$

where M denotes the mathematical expectation. The matrix σ accompanying the noise term ξ in (1.1) is such that $\sigma\sigma'$, where the prime denotes the transposition, is positive definite. Finally, we assume that the vector function $f(t, x) \in E_n$ is measurable over the set of arguments, has a bounded derivative in x and satisfies the conditions

$$|f(t, x_1)|^2 \leqslant \alpha_2 + \alpha_3 |x_1|^2, \qquad \left|\frac{\partial f(t, x)}{\partial x}\right| \leqslant \alpha_1 \qquad (1.2)$$

Here x and x_1 denote any vectors from E_n, the constants $\alpha_i \geqslant 0$ are specified and $|x|$ denotes the Euclidean norm of the vector x. The control $u(t)$ must be chosen so as to minimize the functional

$$J(u) = M\left[x'(T)\,L_1 x(T) + \int_0^T (x'(s)\,L_2(s)\,x(s) + u'(s)\,L_3(s)\,u(s))\,ds\right] \qquad (1.3)$$

where L_i are matrices which are specified and have measurable bounded elements, and L_1, L_2 are negative definite, while $L_3(t)$ is uniformly positive definite on the segment $[0, T]$.

Under the above assumptions and with the control $u(t) = 0$ operating, a solution of the problem (1.1) exists and is unique [6]. On the other hand, if $\varepsilon = 0$, then the solution of the problem of optimal control (1.1), (1.3) can be obtained in its explicit analytic form.

Let us cite some formulas [7] which will be needed in what follows. When $\varepsilon = 0$, the optimal control $u_0(t)$ is

$$u_0(t) = -L_3^{-1} B' P(t)\,x(t) \qquad (1.4)$$

where the nonnegative definite square $n \times n$ matrix $P(t)$ is a solution of the equation

$$P^{\cdot}(t) = -P(t)\,A(t) - A'(t)\,P(t) - L_2(t) + \qquad (1.5)$$
$$P(t)\,B(t)\,L_3^{-1}(t)\,B'(t)\,P(t) = 0, \quad P(T) = L_1$$

2. Let us denote by $V(t, x)$ the Bellman function of the problem (1.1), (1.3). According to [8] this function represents the unique solution of the Cauchy problem

$$LV + \varepsilon f' \frac{\partial V}{\partial x} - \frac{1}{4}\left(\frac{\partial V}{\partial x}\right) B_1\left(\frac{\partial V}{\partial x}\right) + x' L_2 x = 0 \qquad (2.1)$$
$$L = \frac{\partial}{\partial t} + \frac{1}{2}\operatorname{Tr}\left(\sigma\sigma' \frac{\partial^2}{\partial x^2}\right), \quad B_1 = BL_3^{-1}B', \quad V(T, x) = x' L_1 x$$

Here and in what follows we can set $A = 0$ without any loss of generality. The optimal control $v(t, x)$ now becomes

$$v(t, x) = -{}^1/_2 L_3^{-1}(t) B'(t) \partial V(t, x) / \partial x \tag{2.2}$$

In particular, when $\varepsilon = 0$, the Bellman function $V_0(t, x)$ becomes, by virtue of [7],

$$V_0(t, x) = x'P(t)x \tag{2.3}$$

Let us now give the algorithm for constructing the i-th approximation to the optimal control. We write $V(t, x)$ in the form

$$V(t, x) = \sum_{j=0}^{i} \varepsilon^j V_j(t, x) + r_i(t, x) \tag{2.4}$$

Here V_0 is given by (2.3), the equations for V_j are obtained by substituting (2.4) into (2.1) and equating the terms accompanying ε^j and not containing r_i, and all the remaining terms determine the equation for r_i. Let us assume that the i-th approximation $u_i(t, x)$ to the optimal control (2.2) is given

$$u_i(t, x) = -\frac{1}{2} L_3^{-1}(t) B'(t) \sum_{j=0}^{i} \varepsilon^j \frac{\partial V_j}{\partial x} \tag{2.5}$$

To substantiate the formula (2.5) we must show that the difference $V(0, a_0) - J(u_i)$ is of the order of ε^{i+1}. This is proved for various values of i according to a single scheme consisting of constructing two solutions of the system (1.1), one for the control (2.5) and the other for the control (2.2), (2.4), and estimating the moment of the difference between these processes.

We shall elucidate this in more detail using the zero approximation. In this case the equation for r_0 implies that $r_0(t, x)$ is the Bellman function of the following control problem: to find a control ω minimizing the functional

$$M = \int_0^T [2\varepsilon y'(t) P(t) f(t, y(t)) + \omega'(t) L_3(t) \omega(t)] dt \tag{2.6}$$

on the trajectories $y(t)$ of the system of stochastic differential Ito equations

$$dy(t) = [\varepsilon f(t, y(t)) - B_1 P y(t) + B\omega] dt + \sigma d\xi(t) \tag{2.7}$$

The existence of a solution of the control problem (2.7), (2.6) follows from the existence of the solution of the Bellman equation (2.5). Let us denote by $\omega_0(t, y)$ the optimal control in the problem (2.6), (2.7). By virtue of (2.6) and (2.7) [8] a matrix $W(t)$ bounded on $[0, T]$ exists such that

$$\frac{\partial r_0(t_0, x_0)}{\partial x} = M \int_{t_0}^{T} W(t) \frac{\partial}{\partial y} [2\varepsilon y'(t, \omega_0) f(t, y(t, \omega_0))] dt$$

From this and from [9] it follows that

$$|\partial r_0(t, x) / \partial x| \leqslant \varepsilon c (1 + |x|) \tag{2.8}$$

Thus the optimal control v in the problem (1.1), (1.3) differes from the control (1.4) by a quantity which increases at $|x| \to \infty$ not faster than a linear function of $|x|$. Therefore, taking into account (2.8) and [6] we conclude that

$$|J(u_0) - J(v)| \leqslant \varepsilon c (1 + |a_0|^2) \tag{2.9}$$

Note. Let us substitute $x(t, u_0)$ into the control (1.4), and $x(t, v)$ into the optimal control, and find the difference between the resulting quantities. Then, following (2.9) we can conclude that the mathematical expectation of the square of this difference is of the order of ε^2, i.e. in this sense the difference between the optimal control (2.7) and the approximate control (1.4) is of the order of ε^2.

3. The higher order approximations to the optimal control are determined by the formula (2.5). From (2.4) and (2.1) it is obvious that all $V_i\; i \geqslant 1$ are determined in quadratures in terms of the function f, of the previous approximations $V_j,\; j \leqslant i-1$ and of the fundamental solution $\alpha(t, x, \tau, y)$ of the following linear homogeneous parabolic equation:

$$L_0 q = 0, \qquad L_0 = L - x'PB_1 \frac{\partial}{\partial x}$$

We can easily show that

$$\alpha(t, x, \tau, y) = \frac{1}{\sqrt{(2\pi)^n \Delta}} \exp\left\{-\frac{1}{2}(y-m)' D^{-1}(y-m)\right\} \tag{3.1}$$

$$m = z_1(t, \tau), \qquad D = \int_t^\tau z_1(s, \tau)\,\sigma\sigma' z_1'(s, \tau)\,ds$$

where z_1 is the fundamental solution of (2.7) with $\varepsilon = B = \sigma = 0$ and Δ is the determinant of the matrix D.

Validity of the higher order approximations to the optimal approximation is substantiated in the same manner as the validity of the zero approximation (1.4). We must however also show that $\partial V_i / \partial x$ increases with $|x| \to \infty$ not faster than a linear function of $|x|$ and that $\partial^2 V / \partial x^2$ increases not faster than some power of $|x|$. These estimates are obtained for all approximations in the same manner, therefore we shall prove their validity for V_1 only.

The following equation for the determination of V_1 is obtained from (2.4):

$$L_0 V_1 + 2f'Px = 0, \qquad V_1(T, x) = 0 \tag{3.2}$$

From this we can deduce as we did for (2.8), that $\partial V_1 / \partial x$ increases as some linear function of $|x|$. In obtaining an estimate for $\partial^2 V_1 / \partial x^2$ we note that by (3.2) and (3.1) the function V_1 increases with $|x| \to \infty$ not faster than a polynomial of second degree in $|x|$. Therefore, taking into account the inequality given in [10], chapt. 3, we have

$$|\partial^2 V_1(t, x) / \partial x^2| \leqslant c\,[1 + |x|^2]$$

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STABILITY OF RELATIVE EQUILIBRIUM OF A BODY IN A PERTURBED CIRCULAR ORBIT

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We study the stability of relative equilibrium of a body whose center of mass describes a non-Keplerian circular orbit without a center of attraction, under the action of perturbing or controlling forces. The problem is solved under a restricted formulation, in which only the gravitational moments relative to the central field are taken into account. Sufficient conditions of stability of the positions of equilibrium obtained are found, using the Routh theorem in the manner analogous to that developed in [1].

The problem of relative equilibrium of a rigid body in a circular unperturbed orbit and its stability, were investigated recently in detail by many authors [1, 2]. We find however, that in certain concrete problems of celestial mechanics and dynamics of space flights there is a need to generalize this problem to the case of perturbed circular orbits which are realized when perturbing or controlling forces are present. An example of such a problem is the case of circular non-Keplerian orbits in the gravitational field of an axisymmetric planet. The author of [3] has proved, in particular, the existence of circular orbits the plane of which is parallel to the equatorial plane of the planet.

An interesting problem from the point of view of space dynamics is that of setting a synchronous stationary satellite (SS) at an arbitrary latitude. The authors of [4, 5] studied this problem for the center of mass of the satellite, and [6] dealt with the translational-rotational motion of such a satellite under the assumption that an additional constant reactive force is applied to its center of mass. It was shown that a relative equilibrium of a body is possible when its center of mass is in circular motions in a plane that does not contain the center of attraction. This problem was not previously investigated.

The aim of the present paper is to obtain sufficient conditions for the relative equilibrium of a rigid body when the formulation of the problem is restricted, i.e. under the assumption that the motion of the body relative to the center of mass does not influence